

Hardy-Carleman Type Inequalities for Dirac Operators

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Abstract

General Hardy-Carleman type inequalities for Dirac operators are proved. New inequalities are derived involving particular traditionally used weight functions. In particular, a version of the Agmon inequality and Treve type inequalities are established. The case of a Dirac particle in a (potential) magnetic field is also considered. The methods used are direct and based on quadratic form techniques.

Keywords: Spectral theory; Dirac operators; Weighted inequalities.

1 Introduction

In a recent work [DES00] (see also [DEDV07] and [DELV04]) of J. Dolbeault et al. it was proved a version of Hardy type inequality related to the Dirac operator describing the behaviour of a spin 1/2 particle with non-zero rest mass under the influence of an electrostatic potential q . Namely, for a given scalar potential q satisfying

$$q(x) \rightarrow 0, \quad |x| \rightarrow \infty, \\ -\frac{\nu}{|x|} - c_1 \leq q(x) \leq c_2 = \sup_{x \in \mathbb{R}^3} q(x)$$

with (a parameter) $\nu \in (0, 1)$ and $c_1, c_2 \in \mathbb{R}$, there exists a value μ in the interval $(-1, 1)$ such that the inequality

$$\int_{\mathbb{R}^3} q |\varphi|^2 dx \leq \int_{\mathbb{R}^3} \left(\frac{|\sigma \cdot \nabla \varphi|^2}{1 + \mu + q} + (1 - \mu) |\varphi|^2 \right) dx \quad (1.1)$$

holds true for all functions φ in the Sobolev space $W_2^1(\mathbb{R}^3; \mathbb{C}^2)$. In (1.1) $\nabla \varphi$ denotes the distributional gradient of φ , $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ being the triplet of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

μ is in fact taken as the smallest eigenvalue of the Dirac operator

$$H = -i\alpha \cdot \nabla + \beta + q,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and the matrices $\beta, \alpha_j \in M_{4 \times 4}(\mathbb{C}), j = 1, 2, 3$, are defined as

$$\sigma_1 = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \beta = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

(Id is the identity matrix in \mathbb{C}^2). As a consequence, in the case of the Dirac-Coulomb Hamiltonian H , that is, when $q(x) \approx 1/|x|$, a simple limiting argument yields to the following Hardy type inequality

$$\int_{\mathbb{R}^3} \frac{|\varphi|^2}{|x|} dx \leq \int_{\mathbb{R}^3} \left(\frac{|\sigma \cdot \nabla \varphi|^2}{1 + \frac{1}{|x|}} + |\varphi|^2 \right) dx, \quad \varphi \in W_2^1(\mathbb{R}^3; \mathbb{C}^2), \quad (1.2)$$

which can be interpreted also as a *relativistic uncertainly principle* for H .

The mentioned inequalities (1.1) and (1.2) allow in a natural way to describe distinguished self-adjoint extensions of Dirac operators with certain singularities of the potentials [EL07]. It should be mentioned that they are also useful in the study of spectral properties of Dirac operators especially relevant to the problems of scattering theory as, in particular, to get information about the behaviour of the resolvent nearly to the continuous spectrum, in proving of the limiting absorption principle and others.

Our main purposes is to prove Hardy type inequalities for Dirac operators in more general setting involving arbitrary weight functions. By a Dirac operator we mean a first order partial differential operator with constant coefficients of the form

$$H = \sum_{j=1}^n \alpha_j D_j + \beta, \quad (1.3)$$

where $D_j = -i\partial/\partial x_j$ ($j = 1, \dots, n$), $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, α_j ($j = 1, \dots, n$) and β are $m \times m$ Hermitian matrices which satisfy the Clifford's anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \quad (j, k = 1, \dots, n), \quad \alpha_j \beta + \beta \alpha_j = 0 \quad (j = 1, \dots, n), \quad \beta^2 = 1, \quad (1.4)$$

$m = 2^{n/2}$ for n even and $m = 2^{(n+1)/2}$ for n odd; δ_{jk} denotes the Kronecker symbol ($\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$). The Dirac operator H is usually considered acting in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ defined on its maximal domain the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$. However, to cover some more general situations, also important in applications or by themselves, it will be also assumed that the operator H is defined on an arbitrary (open) domain Ω in \mathbb{R}^n . In those cases as a domain of H is certainly taken the Sobolev space of functions defined on Ω .

Our aim is to describe conditions on the weight functions a, b under which an inequality of the form

$$c\|au\| \leq \|bHu\|, \quad u \in \mathcal{D}, \quad (1.5)$$

holds true for a suitable class of functions $u \in \mathcal{D}$, c being a positive constant depending only on a, b and, eventually, Ω . We assume that a and b are sufficiently smooth functions, it will be enough to consider a, b to be of class C^2 .

In (1.5) $\|\cdot\|$ designates the norm on $L_2(\mathbb{R}^n; \mathbb{C}^m)$. We call estimates like that in (1.5) as Hardy-Carleman type inequalities. Inequalities of this kind related mostly to the Laplace operator are also named weighted Hardy inequalities or weighted Poincaré-Sobolev inequalities or weighted Friedrichs inequalities as well. We emanate the classical Hardy inequality for Dirichlet form (see, for instance, [Maz85] or [Dav99] and [KMP07] for a history on the subject and farther references), and the remarkable inequality due to Carleman stated in [Car39] in connection with the unique continuation property for second order elliptic differential equations. Apart from the already mentioned works (there is a vast literature on the topic), we refer to [ABG82] (see also [ABG87]), [Hör07], [Hör83], [Jer86], [JK85], and the references quoted there.

In spite of the fact that the Laplace and Dirac operators are closely connected with each other, however, they behave quite differently, and the methods properly for the Laplace operator are no longer applicable to the Dirac operators case. Nevertheless, as is shown in this paper, some of traditional methods can be refined to be available also for Dirac operators. Compared to the Laplace operator case, the subject concerning Dirac operators, to the best of our knowledge, has been studied rather sparingly in the literature. We mention the work [Jer86] in which certain Carleman inequalities for the (massless) Dirac operator are established.

The paper is organised as follows. In Section 2 we discuss general weight inequalities for the Dirac operator defined by (1.3) and (1.4). Conditions on the weight functions a, b are given in order that an inequality of the form (1.5) hold true. The proofs are based on quadratic form techniques. In Section 3 there is established a Carleman inequality for the particular case, but useful in applications, of radial weight functions. Section 4 contains concrete Carleman type inequalities that are derived from general results by handling special traditionally used weight functions. In this way a version of the Agmon inequality and Treve type inequalities are obtained as particular cases of the general inequalities. In Section 5 we prove an inequality with power like weight functions for which the approach applied previously is not available. The arguments in the proof of the corresponding Hardy-Carleman inequality use eigenfunctions expansions by involving spherical harmonic functions. Finally, in Section 6 the results are extended to the case of the Dirac operator describing a relativistic particle in a potential magnetic field.

2 General Hardy-Carleman inequalities

In this section we discuss general weight inequalities of the form (1.5). In order to make use our method for the proof it is always required that the weight functions a, b to be of class C^2 . We describe conditions under which (1.5) hold true for a suitable class of functions $u \in \mathcal{D}$.

It will be convenient to work in polar coordinates $(r, \omega) \in (0, \infty) \times S^{n-1} : r = |x|$, $\omega = x/|x|$ for $x \neq 0$. Denoting $\omega_j = x_j/|x|$ ($j = 1, \dots, n$), the coordinates of ω , we have

$$\frac{\partial}{\partial x_j} = \omega_j \frac{\partial}{\partial r} + r^{-1} \Omega_j \quad (j = 1, \dots, n),$$

where Ω_j is a vector-field on the unit sphere S^{n-1} satisfying

$$\sum_{j=1}^n \omega_j \Omega_j = 0, \quad \sum_{j=1}^n \Omega_j \omega_j = r \sum_{j=1}^n \frac{\partial \omega_j}{\partial x_j} = n - 1.$$

Let

$$\hat{\alpha} = \sum_{j=1}^n \alpha_j \omega_j,$$

then

$$\begin{aligned} H &= \sum_{j=1}^n \alpha_j \left(-i \left(\omega_j \frac{\partial}{\partial r} + r^{-1} \Omega_j \right) \right) + \beta = \\ &= -i \sum_{j=1}^n \alpha_j \omega_j \frac{\partial}{\partial r} - i r^{-1} \sum_{j=1}^n \alpha_j \Omega_j + \beta = \\ &= -i \hat{\alpha} \frac{\partial}{\partial r} - i r^{-1} \sum_{j=1}^n \alpha_j \Omega_j + \beta, \end{aligned}$$

i.e.,

$$H = -i \hat{\alpha} \frac{\partial}{\partial r} - i r^{-1} \sum_{j=1}^n \alpha_j \Omega_j + \beta. \quad (2.1)$$

It is easy to see that

$$\hat{\alpha}^2 = \left(\sum_{j=1}^n \alpha_j \omega_j \right)^2 = \sum_{j=1}^n \sum_{k=1}^n \alpha_j \alpha_k \omega_j \omega_k = 1,$$

and

$$\begin{aligned} \hat{\alpha} \sum_{j=1}^n \alpha_j \Omega_j &= \sum_{j=1}^n \sum_{k=1}^n \alpha_j \alpha_k \omega_j \Omega_k = \\ &= \sum_{j=1}^n \alpha_j^2 \omega_j \Omega_j + \sum_{j \neq k} \alpha_j \alpha_k \omega_j \Omega_k = \sum_{j=1}^n \omega_j \Omega_j + \sum_{j < k} \alpha_j \alpha_k \omega_j \Omega_k + \sum_{j > k} \alpha_j \alpha_k \omega_j \Omega_k = \\ &= \sum_{j < k} \alpha_j \alpha_k \omega_j \Omega_k + \sum_{j < k} (-\alpha_j \alpha_k) \omega_k \Omega_j = \\ &= \sum_{j < k} \alpha_j \alpha_k (\omega_j \Omega_k - \omega_k \Omega_j). \end{aligned}$$

We let

$$L = \sum_{j < k} \alpha_j \alpha_k (\omega_j \Omega_k - \omega_k \Omega_j),$$

an operator acting only in the ω variables, then

$$i \hat{\alpha} H = i \hat{\alpha} \left(-i \hat{\alpha} \frac{\partial}{\partial r} - i r^{-1} \sum_{j=1}^n \alpha_j \Omega_j + \beta \right) =$$

$$= \frac{\partial}{\partial r} + r^{-1}L + i\hat{\alpha}\beta,$$

i.e.,

$$i\hat{\alpha}H = \frac{\partial}{\partial r} + r^{-1}L + i\hat{\alpha}\beta. \quad (2.2)$$

Note that

$$L = \sum_{j < k} \alpha_j \alpha_k \left(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right),$$

that follows immediately from the relations

$$\begin{aligned} x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} &= x_j \left(\omega_k \frac{\partial}{\partial r} + r^{-1} \Omega_k \right) - x_k \left(\omega_j \frac{\partial}{\partial r} + r^{-1} \Omega_j \right) = \\ &= r \omega_j \omega_k \frac{\partial}{\partial r} + \omega_j \Omega_k - r \omega_k \omega_j \frac{\partial}{\partial r} - \omega_k \Omega_j = \\ &= \omega_j \Omega_k - \omega_k \Omega_j. \end{aligned}$$

Returning to the inequality (1.5) we put

$$v = bu$$

and observe that for any smooth function φ one has

$$[H, \varphi] = -i \sum_{j=1}^n \alpha_j \frac{\partial \varphi}{\partial x_j}, \quad (2.3)$$

where $[H, \varphi]$ denotes the commutator of H and φ , viewing simultaneously φ as the multiplication operator by the function φ .

Using the obtained relation (2.3) we can write

$$bH = Hb + i \sum_{j=1}^n \alpha_j \frac{\partial b}{\partial x_j}$$

or

$$bHb^{-1} = H + iB,$$

where

$$B = b^{-1} \sum_{j=1}^n \alpha_j \frac{\partial b}{\partial x_j}. \quad (2.4)$$

Thus, the inequality (1.5) reduces to the estimation from the below of the quadratic form

$$h[v] = \|(H + iB)v\|^2$$

on suitable elements v .

Let us start with the case of the Dirac operator considered on \mathbb{R}^n . Using the polar coordinates, by (2.1) and (2.2), $h[v]$ can be represented as follows

$$h[v] = \int_0^\infty \int_{S^{n-1}} r^{n-1} \left| \left(-i\hat{\alpha} \frac{\partial}{\partial r} - ir^{-1} \sum_{j=1}^n \alpha_j \Omega_j + \beta + iB(r\omega) \right) v(r\omega) \right|^2 dr d\omega =$$

$$\begin{aligned}
&= \int_0^\infty \int_{S^{n-1}} r^{n-1} \left| i\hat{\alpha} \left(-i\hat{\alpha} \frac{\partial}{\partial r} - ir^{-1} \sum_{j=1}^n \alpha_j \Omega_j + \beta + iB(r\omega) \right) v(r\omega) \right|^2 dr d\omega = \\
&= \int_0^\infty \int_{S^{n-1}} r^{n-1} \left| \left(\frac{\partial}{\partial r} + r^{-1}L + i\hat{\alpha}\beta - \hat{\alpha}B(r\omega) \right) v(r\omega) \right|^2 dr d\omega.
\end{aligned}$$

Letting $r = e^t$ we have

$$\frac{\partial}{\partial r} = e^{-t} \frac{\partial}{\partial t},$$

and then

$$h[v] = \int_{-\infty}^\infty \int_{S^{n-1}} e^{t(n-2)} \left| \left(\frac{\partial}{\partial t} + L + i\hat{\alpha}\beta e^t - \hat{\alpha}B(e^t\omega) e^t \right) v(e^t\omega) \right|^2 dt d\omega.$$

In order to remove the exponent in the expression on right-hand side, we let

$$\tilde{v}(t, \omega) = e^{\nu t} v(e^t\omega) \quad \text{with} \quad 2\nu = n - 2$$

and, taking into account that

$$\left(\frac{\partial}{\partial t} \right) e^{-\nu t} = -\nu e^{-\nu t} + e^{-\nu t} \frac{\partial}{\partial t} = e^{-\nu t} \left(\frac{\partial}{\partial t} - \nu \right),$$

we obtain

$$h[v] = \int_{-\infty}^\infty \int_{S^{n-1}} \left| \left(\frac{\partial}{\partial t} - \frac{n-2}{2} + L + i\hat{\alpha}\beta e^t - \hat{\alpha}B(e^t\omega) e^t \right) \tilde{v}(t, \omega) \right|^2 dt d\omega.$$

It follows

$$\begin{aligned}
h[v] &= \int_{-\infty}^\infty \int_{S^{n-1}} \left| \frac{\partial}{\partial t} \tilde{v}(t, \omega) \right|^2 dt d\omega + \\
&+ \int_{-\infty}^\infty \int_{S^{n-1}} \left| \left(L + i\hat{\alpha}\beta e^t - \hat{\alpha}B(e^t\omega) e^t - \frac{(n-2)}{2} \right) \tilde{v}(t, \omega) \right|^2 dt d\omega + \\
&+ \int_{-\infty}^\infty \int_{S^{n-1}} \left\langle \left(-i\hat{\alpha}\beta e^t + \hat{\alpha} \frac{\partial (B(e^t\omega) e^t)}{\partial t} \right) \tilde{v}(t, \omega), \tilde{v}(t, \omega) \right\rangle dt d\omega
\end{aligned}$$

($\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{C}^m).

In this way we obtain the following estimation

$$h[v] \geq \int_{-\infty}^\infty \int_{S^{n-1}} \langle G(t, \omega) \tilde{v}(t, \omega), \tilde{v}(t, \omega) \rangle dt d\omega, \quad (2.5)$$

where

$$G(t, \omega) = -i\hat{\alpha}\beta e^t + \hat{\alpha} \frac{\partial (B(e^t\omega) e^t)}{\partial t}.$$

The form on the right-hand side of (2.5) can be further transformed as follows (note that $\partial/\partial t = r\partial/\partial r$)

$$\int_{-\infty}^\infty \int_{S^{n-1}} \langle G(t, \omega) \tilde{v}(t, \omega), \tilde{v}(t, \omega) \rangle dt d\omega =$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{S^{n-1}} e^{2\nu t} \langle G(t, \omega) v(e^t \omega), v(e^t \omega) \rangle dt d\omega = \\
&= \int_0^{\infty} \int_{S^{n-1}} r^{n-2} \left\langle \left(-i\hat{\alpha}\beta r + \hat{\alpha} r \frac{\partial(B(r\omega)r)}{\partial r} \right) v(r\omega), v(r\omega) \right\rangle r^{-1} dr d\omega = \\
&= \int_0^{\infty} \int_{S^{n-1}} r^{n-1} \left\langle r^{-1} \left(-i\hat{\alpha}\beta + \hat{\alpha} \frac{\partial(B(r\omega)r)}{\partial r} \right) b(r\omega)^2 u(r\omega), u(r\omega) \right\rangle dr d\omega.
\end{aligned}$$

Now, wishing to involve the weight function a we require that the matrix-valued function

$$M(r, \omega) = r^{-1} \left(-i\hat{\alpha}\beta + \hat{\alpha} \frac{\partial(B(r\omega)r)}{\partial r} \right) b(r\omega)^2 a(r\omega)^{-2} \quad (2.6)$$

is positive definite (with respect to the quadratic forms) uniformly with respect to r and ω , that in fact means

$$\langle M(r, \omega) u(r\omega), u(r\omega) \rangle \geq c |u(r\omega)|^2 \quad (2.7)$$

for a positive constant c independent of u, r and ω . Under this condition there holds

$$h[v] \geq c \int_0^{\infty} \int_{S^{n-1}} r^{n-1} |a(r\omega) u(r\omega)|^2 dr d\omega = c \int_{\mathbb{R}^n} |a(x) u(x)|^2 dx,$$

that leads to the desired inequality (1.5). For, as is seen, \mathcal{D} can be taken the set of all functions u belonging to the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ having compact support in $\mathbb{R}^n \setminus \{0\}$.

We summarize the above discussion in the following theorem.

Theorem 2.1. *Let a, b be positive functions of the class C^2 for which the condition (2.7) is fulfilled. Then for any function u belonging to the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ and having a compact support in the set $\mathbb{R}^n \setminus \{0\}$ the inequality (1.5) holds true with a positive constant c depending only on a and b .*

Remark 2.2. The above arguments remain valid for the Dirac operator H considered on an arbitrary open domain Ω . The functions u in (1.5) must be then taken in the Sobolev space $W_2^1(\Omega; \mathbb{C}^m)$ having compact supports in the set Ω , respectively, in $\Omega \setminus \{0\}$ if $0 \in \Omega$. Incidentally, the condition (2.7) should be fulfilled on Ω . Note that the condition (2.7) can be interpreted as that determining Ω on which the inequality (1.5) holds true.

The inequalities like one as in (1.5) can be called *general Hardy-Carleman inequalities* or, simply, Carleman type Inequalities. Various Carleman type inequalities can be derived by choosing suitable weight functions a, b defined on a domain Ω in \mathbb{R}^n . In the remainder of this section we confine ourselves to make some remarks still concerning on general situations of Carleman inequalities. Concrete Carleman type inequalities and further remarks will be given in the next sections.

Let the weight functions a, b be chosen satisfying

$$b(x) = |x|^{1/2} a(x), \quad x \in \Omega \quad (2.8)$$

for a given (open) domain Ω in \mathbb{R}^n . Then

$$M(r, \omega) = -i\hat{\alpha}\beta + \hat{\alpha}\frac{\partial(B(r\omega)r)}{\partial r} \quad (2.9)$$

and, in order to establish a Carleman inequality for this case, we have to look that the matrix-valued function $M(r, \omega)$ given by (2.9) to be positive definite uniformly on Ω . To this end it should be noted that the matrix $-i\hat{\alpha}\beta$ is symmetric and has only two eigenvalues ± 1 (the point is that, additionally, $(-i\hat{\alpha}\beta)^2 = 1$ and $-i\hat{\alpha}\beta \neq \pm 1$). It is clear that the condition (2.7) will be fulfilled if the eigenvalues of the matrix

$$M_0(r, \omega) = \hat{\alpha}\frac{\partial(B(r\omega)r)}{\partial r}$$

are situated on the right-side of 1, and if

$$d = \inf_{r, \omega} (\lambda_{\min}(r, \omega) - 1) > 0, \quad (2.10)$$

where $\lambda_{\min}(r, \omega)$ is the least of eigenvalues of $M_0(r, \omega)$. Obviously, $\lambda_{\min}(r, \omega)$ can be chosen depending continuously on r and ω , provided that the function b is of the class C^2 .

Thus, we can formulate the following.

Corollary 2.3. *Let a, b be functions of the class C^2 satisfying (2.8) on a given open domain Ω in \mathbb{R}^n , and suppose that the condition (2.10) is fulfilled. Then, for any function u in the Sobolev space $W_2^1(\Omega; \mathbb{C}^m)$ having its compact support in the set $\Omega \setminus \{0\}$, there holds the following inequality*

$$c \int_{\Omega} |a(x)u(x)|^2 dx \leq \int_{\Omega} |x| |a(x)Hu(x)|^2 dx \quad (2.11)$$

with a positive constant c which can be taken equal to d (d being defined by (2.10)).

Remark 2.4. In case the domain Ω is bounded the factor $|x|$ in (2.11) can be omitted by changing suitably the constant c .

3 The case of radial weight functions

Throughout this section we suppose that the weight functions a, b depend only on the radial coordinate r , $r = |x|$. For this case the conditions like (2.7), being crucial for the fulfilment of a desired Hardy-Carleman type inequality, became considerably simpler. So, if $b = b(r)$ depends only on the radial coordinate r , then

$$\frac{\partial b}{\partial x_j} = b'(r) \frac{\partial r}{\partial x_j} = b'(r) \omega_j,$$

and, by (2.4), one has

$$B(r) = b(r)^{-1} \sum_{j=1}^n \alpha_j b'(r) \omega_j = b(r)^{-1} b'(r) \hat{\alpha},$$

i.e.,

$$B(r) = b(r)^{-1}b'(r)\hat{\alpha}.$$

Hence,

$$M_0(r, \omega) = \hat{\alpha} \frac{\partial(B(r, \omega)r)}{\partial r} = (b(r)^{-1}b'(r)r)',$$

i.e.,

$$M_0(r, \omega) = (b(r)^{-1}b'(r)r)',$$

Thus the matrix-valued function $M_0(r, \omega)$ reduces in fact to a scalar function depending only on r . If, in addition, the weight functions are connected between themselves by the relation (2.8), then the condition (2.10) becomes as follows

$$c := \inf_r ((b(r)^{-1}b'(r)r)' - 1) > 0, \quad (3.1)$$

which, certainly, ensures the fulfilment of the inequality (1.5) with the constant c determined as above.

A particular case of just mentioned inequality can be obtained by taking

$$a(x) = |x|^{-1/2}e^{\tau\varphi(x)} \quad \text{and} \quad b(x) = e^{\tau\varphi(x)}, \quad x \neq 0$$

with $\tau > 0$ as a parameter, and φ being a function of the class C^2 depending only on the radial coordinate r . For this case we have

$$\begin{aligned} (b(r)^{-1}b'(r)r)' &= (e^{-\tau\varphi(r)}\tau\varphi'(r)e^{\tau\varphi(r)}r)' = \\ &= \tau(\varphi'(r)r)' = \tau(\varphi''(r)r + \varphi'(r)), \end{aligned}$$

i.e.,

$$(b(r)^{-1}b'(r)r)' = \tau(\varphi''(r)r + \varphi'(r)).$$

Now, it is clear that the condition (3.1) is verified by assuming $\tau\gamma - 1 > 0$, where

$$\gamma = \inf(\varphi''(r)r + \varphi'(r)) > 0 \quad (\text{on } \Omega) \quad (3.2)$$

Thus, the following assertion can be made.

Theorem 3.1. *Let φ be a function of class C^2 depending only on the radial coordinate r and satisfying (3.2) on a given open domain Ω in \mathbb{R}^n . Then, for any function u in the Sobolev space $W_2^1(\Omega; \mathbb{C}^m)$ having its compact support in $\Omega \setminus \{0\}$, the following Carleman type inequality*

$$c \int_{\Omega} |x|^{-1} e^{2\tau\varphi(x)} |u(x)|^2 dx \leq \int_{\Omega} e^{2\tau\varphi(x)} |Hu(x)|^2 dx$$

holds true for $\tau > \gamma^{-1}$ (in particular, for sufficiently large τ) and a positive constant c depending only on τ and φ .

4 Example of Hardy-Carleman inequalities

In this section we derive concrete Hardy-Carleman inequalities by handling special frequently encountered weight functions. We restrict ourselves to consider the Dirac operator describing a relativistic particle with negligible mass. In this case the term H containing β is absent. In order to distinguish this special case the Dirac operator will be denoted by H_0 , so

$$H_0 = \sum_{j=1}^n \alpha_j D_j$$

with all attributed conditions as in general case. For the sake of simplicity, in what follows, we will always consider the operator H_0 on the whole space \mathbb{R}^n , that is acting in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ on its domain the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$.

Hardy-Carleman type inequalities will be established for the operator H_0 by choosing suitable weight functions depending only on radial coordinate r . Under the hypotheses made above the matrix-valued function $M(r, \omega)$, as it was already mentioned before, reduces to a scalar function depending only on r , we denote it by $M(r)$. Namely (cf. Section 3),

$$M(r) = r^{-1}(b(r)^{-1}b'(r)r)'b(r)^2a(r)^{-2}.$$

Example 4.1. Now, letting

$$b(x) = (1 + |x|^2)^{\tau/2}, \quad \tau > 0,$$

we have

$$r^{-1}(b(r)^{-1}b'(r)r)'b(r)^2 = 2\tau(1 + r^2)^{\tau-2},$$

If we take

$$a(r) = (1 + r^2)^{(\tau-2)/2}, \quad r > 0,$$

we obtain that

$$M(r) = 2\tau > 0.$$

Thus we have proved the following inequality

$$2\tau \int_{\mathbb{R}^n} (1 + |x|^2)^{\tau-2} |u(x)|^2 dx \leq \int_{\mathbb{R}^n} (1 + |x|^2)^\tau |H_0 u(x)|^2 dx, \quad (4.1)$$

for all $\tau > 0$.

In (4.1) and in all considered further inequalities as well, it is assumed that the function u belongs to the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ having compact support in the set $\mathbb{R}^n \setminus \{0\}$.

The following particular cases, namely

$$2 \int_{\mathbb{R}^n} (1 + |x|^2)^{-1} |u(x)|^2 dx \leq \int_{\mathbb{R}^n} (1 + |x|^2) |H_0 u(x)|^2 dx, \quad (4.2)$$

for $\tau = 1$, and

$$4 \int_{\mathbb{R}^n} |u(x)|^2 dx \leq \int_{\mathbb{R}^n} (1 + |x|^2) |H_0 u(x)|^2 dx, \quad (4.3)$$

for $\tau = 2$, are important in applications and by themselves. The inequality (4.2) can be named as an *Agmon type inequality* (cf. [Agm75]) whereas (4.3) as a *Hardy type inequality* for the Dirac operator H_0 .

Example 4.2. Next, we consider

$$b(x) = e^{\tau|x|^\alpha/2}, \quad \tau > 0, \alpha \in \mathbb{R}, \alpha \neq 0.$$

In this case

$$b(r) = e^{\tau r^\alpha/2}, \quad b'(r) = (\tau/2)\alpha r^{\alpha-1} e^{\tau r^\alpha/2},$$

and

$$r^{-1}(b(r)^{-1}b'(r)r)'b(r)^2 = (\tau/2)\alpha^2 r^{\alpha-2} e^{\tau r^\alpha},$$

from which it is seen that it can be taken

$$a(r) = r^{(\alpha-2)/2} e^{\tau r^\alpha/2}, \quad r > 0.$$

Then

$$M(r) = \tau\alpha^2/2 > 0,$$

and, thus, we obtain the following inequality

$$(\alpha^2\tau/2) \int_{\mathbb{R}^n} |x|^{\alpha-2} e^{\tau|x|^\alpha} |u(x)|^2 dx \leq \int_{\mathbb{R}^n} e^{\tau|x|^\alpha} |H_0 u(x)|^2 dx, \quad (4.4)$$

for $\tau > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$.

The following useful inequality

$$(\tau/2) \int_{\mathbb{R}^n} |x|^{-1} e^{\tau|x|} |u(x)|^2 dx \leq \int_{\mathbb{R}^n} e^{\tau|x|} |H_0 u(x)|^2 dx, \quad (4.5)$$

is a particular case of (4.4) for $\alpha = 1$.

The inequality (4.4) for $\alpha = 2$ corresponds to the following one

$$2\tau \int_{\mathbb{R}^n} e^{\tau|x|^2} |u(x)|^2 dx \leq \int_{\mathbb{R}^n} e^{\tau|x|^2} |H_0 u(x)|^2 dx, \quad (4.6)$$

which can be called as a *Treue type inequality* for the Dirac operator H_0 . We cite [Tre61] for related inequalities involving differential operators.

Example 4.3. Finally, let us consider the weight function

$$b(x) = e^{\tau(\log|x|)^2/2}, \quad \tau > 0.$$

We have

$$b(r) = e^{\tau(\log r)^2/2}, \quad b'(r) = \tau r^{-1}(\log r) e^{\tau(\log r)^2/2},$$

hence

$$r^{-1}(b(r)^{-1}b'(r)r)'b(r)^2 = \tau r^{-2} e^{\tau(\log r)^2}.$$

If it is taken

$$a(r) = r^{-1}e^{\tau(\log r)^2/2},$$

then

$$M(r) = \tau,$$

and, thus, we have proved the following inequality

$$\tau \int_{\mathbb{R}^n} |x|^{-2} e^{\tau(\log |x|)^2} |u(x)|^2 dx \leq \int_{\mathbb{R}^n} e^{\tau(\log |x|)^2} |H_0 u(x)|^2 dx \quad (4.7)$$

for $\tau > 0$.

Remark 4.1. A related inequality to (4.7) was proved in [Jer86] (cf. [Jer86], Theorem 2). However, in [Jer86] instead of the whole space \mathbb{R}^n is taken a domain $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ assuming $0 < a < b < 1$. In [Jer86] it is in fact proved the following one

$$\|e^{\tau\varphi}u\|_{L^q(\Omega;\mathbb{C}^m)} \leq C\|e^{\tau\varphi}H_0u\|_{L^2(\Omega;\mathbb{C}^m)} \quad \text{for all } u \in C_0^\infty(\Omega;\mathbb{C}^m), \quad (4.8)$$

where $\varphi(x) = (\log |x|)^2/2$, $\tau > 0$, $q = (6n - 4)/(3n - 6)$, and C depending only on a, b , and n . By applying our arguments an inequality like (4.8) follows for $q = 2$ as well, but with a constant C depending on a, b, n and also τ .

5 A Carleman type inequality (another approach)

In this section we study the following Carleman type inequality

$$c \int_{\mathbb{R}^n} |x|^\tau |u(x)|^2 dx \leq \int_{\mathbb{R}^n} |x|^{\tau+2} |H_0 u(x)|^2 dx, \quad \tau \in \mathbb{R}, \quad (5.1)$$

for the Dirac operator H_0 . Recall that H_0 denotes the Dirac operator for the case of a particle with negligible mass.

It is easily seen that for the weight functions

$$a(x) = |x|^{\tau/2}, \quad b(x) = |x|^{(\tau+2)/2},$$

as in (5.1), the function $M(r)$, defined as in the previous section, is identically null.

In this connection the results discussed in previous sections cannot be applied to obtain an inequality with such weight functions. On the other hand, it seems that the inequality (5.1) in general fails. The following is true however.

Theorem 5.1. *Let $n > 1$ and let τ be a real number such that $\tau \neq 2k - n$ for integers $k \in \mathbb{Z}$. Then, the inequality (5.1) holds for any function u in the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ having compact support in the set $\mathbb{R}^n \setminus \{0\}$ with a positive constant c depending only on $d := \min_{k \in \mathbb{Z}} |\tau + n - 2k|$.*

Proof. It will be convenient to pass in (5.1) to polar coordinates. We have

$$c \int_0^\infty \int_{S^{n-1}} r^{n-1} r^\tau |u(r\omega)|^2 dr d\omega \leq$$

$$\leq \int_0^\infty \int_{S^{n-1}} r^{n-1} r^{\tau+2} \left| \left(-i\hat{\alpha} \frac{\partial}{\partial r} - i r^{-1} \sum_{j=1}^n \alpha_j \Omega_j \right) u(r\omega) \right|^2 dr d\omega,$$

or equivalently,

$$c \int_0^\infty \int_{S^{n-1}} r^{n-1} r^\tau |u(r\omega)|^2 dr d\omega \leq \int_0^\infty \int_{S^{n-1}} r^{n-1} r^{\tau+2} \left| \left(\frac{\partial}{\partial r} + r^{-1} L \right) u(r\omega) \right|^2 dr d\omega.$$

Further we let $r = e^t$. Then

$$\frac{\partial}{\partial r} = e^{-t} \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial r} + r^{-1} L = e^{-t} \left(\frac{\partial}{\partial t} + L \right),$$

and we have

$$\begin{aligned} & c \int_{-\infty}^\infty \int_{S^{n-1}} e^{(n-1)t} e^{\tau t} |u(e^t \omega)|^2 e^t dt d\omega \leq \\ & \leq \int_{-\infty}^\infty \int_{S^{n-1}} e^{(n-1)t} e^{(\tau+2)t} e^{-2t} \left| \left(\frac{\partial}{\partial t} + L \right) u(e^t \omega) \right|^2 e^t dt d\omega, \end{aligned}$$

i.e.,

$$c \int_{-\infty}^\infty \int_{S^{n-1}} e^{(\tau+n)t} |u(e^t \omega)|^2 dt d\omega \leq \int_{-\infty}^\infty \int_{S^{n-1}} e^{(\tau+n)t} \left| \left(\frac{\partial}{\partial t} + L \right) u(e^t \omega) \right|^2 dt d\omega.$$

To remove the exponents denote

$$v(t, \omega) = e^{\nu t} u(e^t \omega) \quad \text{with} \quad 2\nu = \tau + n.$$

We have

$$\left(\frac{\partial}{\partial t} \right) e^{-\nu t} = -\nu e^{-\nu t} + e^{-\nu t} \frac{\partial}{\partial t} = e^{-\nu t} \left(\frac{\partial}{\partial t} - \nu \right),$$

i.e.,

$$\left(\frac{\partial}{\partial t} \right) e^{-\nu t} = e^{-\nu t} \left(\frac{\partial}{\partial t} - \nu \right),$$

and the inequality becomes

$$c \int_{-\infty}^\infty \int_{S^{n-1}} |v(t, \omega)|^2 dt d\omega \leq \int_{-\infty}^\infty \int_{S^{n-1}} \left| \left(\frac{\partial}{\partial t} - \nu + L \right) v(t, \omega) \right|^2 dt d\omega. \quad (5.2)$$

As is easily seen, it is sufficient to check the obtained inequality for functions of the form

$$v(t, \omega) = f(t) v_k(\omega),$$

where v_k are eigenfunctions (spherical functions) corresponding to the eigenvalues of the operator L , i.e.,

$$L v_k = k v_k.$$

Recall that

$$\sigma(L) \subset \mathbf{Z},$$

and that

$$L(L + n - 2) = -\Delta_\omega,$$

where $-\Delta_\omega$ denotes the Laplace-Beltrami operator of the sphere S^{n-1} . We cite [Ste70] for the details concerning spectral properties of the operator Δ_ω .

It can be supposed that

$$\int_{S^{n-1}} |v_k(\omega)|^2 = 1.$$

Then (5.2) becomes

$$c \int_{-\infty}^{\infty} |f(t)|^2 dt \leq \int_{-\infty}^{\infty} \left| \left(\frac{\partial}{\partial t} - \nu + k \right) f(t) \right|^2 dt. \quad (5.3)$$

In terms of Fourier transform the inequality (4.3) is written as follows

$$c \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi \leq \int_{-\infty}^{\infty} |(i\xi - \nu + k)\widehat{f}(\xi)|^2 d\xi,$$

where

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\xi} dt.$$

The last inequality reduces to the following estimate

$$c \leq |i\xi - \nu + k|^2.$$

It can be taken

$$c \leq (\tau + n - 2k)^2/4,$$

provided that

$$|i\xi - \nu + k|^2 = \xi^2 + (\nu - k)^2 \geq (\nu - k)^2 = (\tau + n - 2k)^2/4.$$

This completes the proof. \square

6 Inequalities for the Dirac operator with a magnetic field

Let H_A denote the Dirac operator with a magnetic field

$$H_A = \sum_{j=1}^n \alpha_j (D_j - A_j(x)) + \beta,$$

where $A(x) = (A_1(x), \dots, A_n(x))$ is a vector potential describing the magnetic field. Assume that A is a smooth vector-valued function with its components A_j sufficiently rapidly decreasing (at infinity) functions in order to preserve the same domain the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ (or, respectively $W_2^1(\Omega; \mathbb{C}^m)$ if it is confined on a domain Ω in \mathbb{R}^n) as for the corresponding free Dirac operator.

For special classes of magnetic fields, but sufficiently large and important for applications, weighted estimates like those discussed in the previous sections, can be reduced to the usual case of the free Dirac operator. So, let the magnetic potential A be of the form

$$A = \nabla \varphi$$

($\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ denotes the gradient operator), where φ is a real-valued function possessing required properties in accordance with those of the magnetic field A .

Now, let an inequality of the form (1.5), i.e.,

$$c \|au\| \leq \|bHu\|, \quad u \in D,$$

holds true. Then there holds the following one

$$c \|e^{i\varphi}au\| \leq \|e^{i\varphi}bHu\|, \quad u \in D,$$

provided that $|e^{i\varphi}| = 1$ (it was supposed that φ is a real-valued function).

Denoting

$$v = e^{i\varphi}u,$$

the last inequality becomes

$$C \|av\| \leq \|b e^{i\varphi} H e^{-i\varphi} v\|$$

for $v = e^{i\varphi}u$ with $u \in D$.

According to the relation (2.3) we can write

$$[H, e^{i\varphi}] = -i \sum_{j=1}^n \left(i e^{i\varphi} \frac{\partial \varphi}{\partial x_j} \right) \alpha_j = e^{i\varphi} \sum_{j=1}^n \left(\frac{\partial \varphi}{\partial x_j} \right) \alpha_j,$$

i.e.,

$$H e^{i\varphi} = e^{i\varphi} H + e^{i\varphi} \sum_{j=1}^n \left(\frac{\partial \varphi}{\partial x_j} \right) \alpha_j,$$

or, what is the same,

$$e^{i\varphi} H e^{-i\varphi} = H - \sum_{j=1}^n \left(\frac{\partial \varphi}{\partial x_j} \right) \alpha_j.$$

But

$$A = \nabla \varphi$$

is equivalent to

$$\sum_{j=1}^n A_j(x) \alpha_j = \sum_{j=1}^n \left(\frac{\partial \varphi}{\partial x_j} \right) \alpha_j.$$

This last fact can be easily explained by using the anticommutation properties of the matrices α_j ($j = 1, \dots, n$).

So,

$$e^{i\varphi} H e^{-i\varphi} = H - \sum_{j=1}^n A_j(x) \alpha_j = H_A,$$

i.e.

$$e^{i\varphi} H e^{-i\varphi} = H_A,$$

and, thus, we obtain an inequality

$$c \|av\| \leq \|b H_A v\| \tag{6.1}$$

for the Dirac operator H_A with the same weight functions a, b and the constant c as in (1.5).

Theorem 6.1. *Under the above hypotheses suppose that the weight functions a, b are of class C^2 satisfying the condition (2.7). Then an inequality (6.1) for the Dirac operator H_A holds true for all functions v belonging to the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$ and having compact supports in the set $\mathbb{R}^n \setminus \{0\}$ with a positive constant c depending only on a and b .*

In view of the discussion undertaken above the other results mentioned previously can be extended in obvious fashion to the case of the Dirac operator H_A .

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